SYNOPSYS®

# A Lens Designer's View of Metaoptics: Aberration Theory for Flat Optics

John R. Rogers, Yijun Ding

June 16, 2025

# Radiative Transfer

Conservation of Energy

© 2025 Synopsys, Inc.

# Radiative Transfer Equation

$$d^2\Phi = L\, cos\theta\; d\Omega\; dA$$

$$\Phi = L \int dA \int cos\theta\; d\Omega$$

Flux

Radiance

Area

Projected
Solid
Angle
(PSA)

For a “spherical cap”
(i.e., a wavefront)

$$PSA = \pi\, sin^2 U$$

© 2025 Synopsys, Inc.

# Radiative Transfer Equation

$$\Phi = \pi L(\,Area\ \times sin^2 U)$$

$$\Phi = \pi \left(\frac{L}{n^2}\right)(n^2 \times Area\ \times sin^2 U)$$

Constant (Conservation of Energy)

Constant

Constant (“Brightness Theorem”)

Constant (Conservation of Étendue)

© 2025 Synopsys, Inc.

# Magnification from Radiative Transfer

We have: $$(n^2 \times Area \ \times sin^2U) \ = \text{constant}$$

In one dimension: $$n \times (Transverse\ dimension) \times sinU = \text{constant}$$

Therefore: $$(Transverse\ dimension) = \left(\frac{\text{constant}}{nsinU}\right)$$

Define: $$M \equiv \frac{(image\ size)}{(object\ size)}$$

Then: $$M = \frac{nsinU}{n'sinU'} = \frac{NA}{NA'}$$

Non-paraxial version of: $$m = \frac{nu}{n'u'}$$

© 2025 Synopsys, Inc.

# Two things we have learned:

Then: $$M = \frac{n sinU}{n' sinU'} = \frac{NA}{NA'}$$

Non-paraxial version of: $$m = \frac{nu}{n'u'}$$

First: the non-paraxial form of magnification involves the *sines* of the ray angles, not the tangents, or the angles converted into radians, or anything else.

# Two things we have learned:

$$M = \frac{n sinU}{n' sinU'} = \frac{NA}{NA'}$$

Non-paraxial version of: $m = \frac{nu}{n'u'}$

First: the non-paraxial form of magnification involves the *sines* of the ray angles, not the tangents, or the angles converted into radians, or anything else.

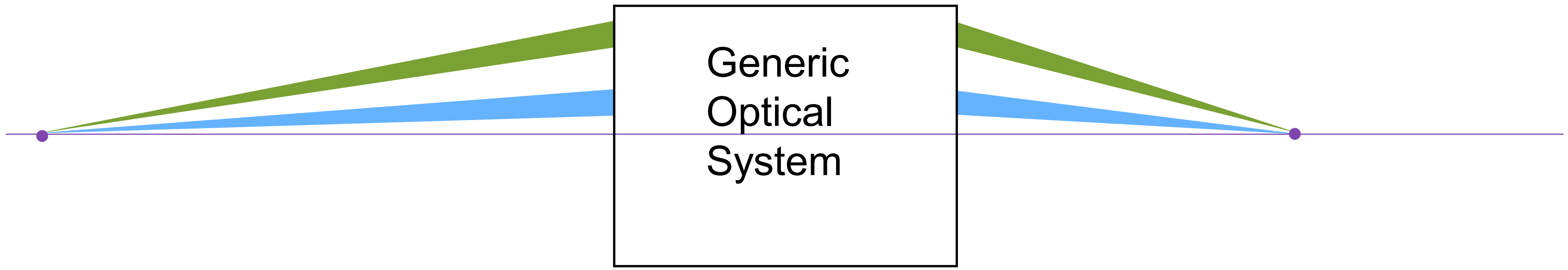
Second: This equation was derived using conservation of energy arguments. It does not depend on the mechanism by which the ray angles are changed. Therefore, it applies to:

- Refractive optics,
- Reflective optics,
- Diffractive optics,
- Meta-optics,
- Magnetic focusing of electron beams,
- Anything else that obeys conservation of energy.

© 2025 Synopsys, Inc.

# Uniform Magnification across the Beam (or Pupil) (Highly Desired!)

- The magnification is the ratio (object space to image space) of the NA’s.
- If we want the magnification to be uniform across the pupil, then we want the ratio of the NA’s for the blue part of the pupil to equal that for the green part of the pupil.

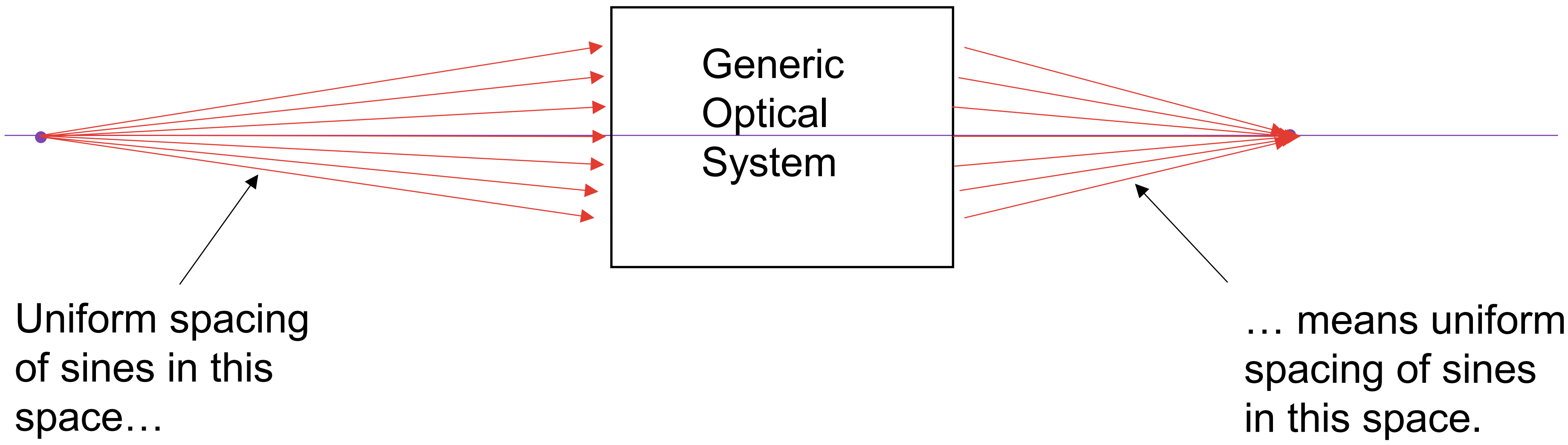



© 2025 Synopsys, Inc. 

# Uniform Magnification across the Beam (or Pupil)

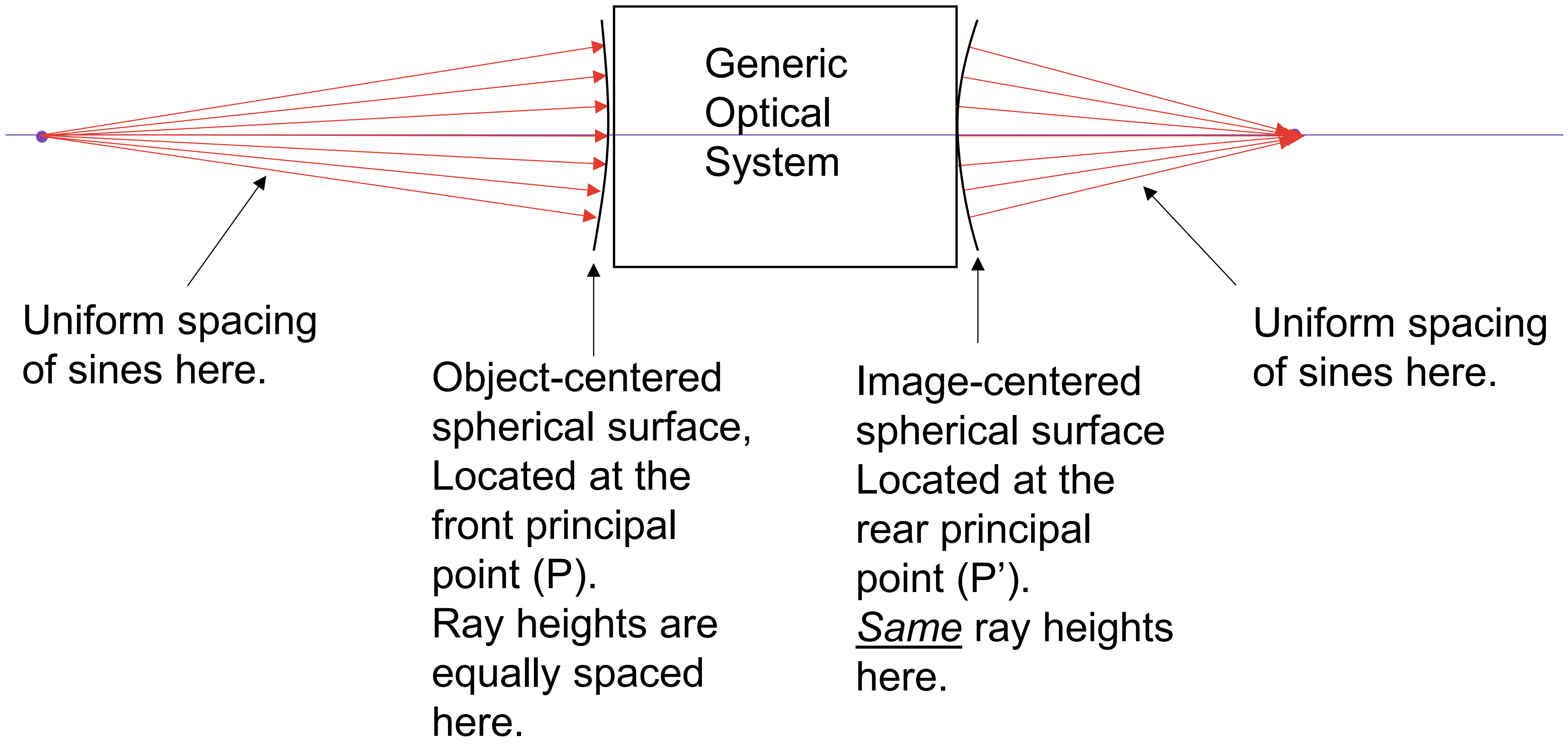
- The magnification is the ratio (object space to image space) of the NA’s.
- Rays that enter the system equally-spaced in NA-space

…should emerge equally-spaced in NA-space (though perhaps covering a different total NA)




© 2025 Synopsys, Inc.

# Uniform Magnification across the Beam (or Pupil)




© 2025 Synopsys, Inc.

# The Abbe Sine Condition

- We can express the desired condition, for every non-paraxial zone of the pupil, as:

$$M_{non-paraxial} = \frac{nSinU}{n'SinU'} = \frac{nu}{n'u'} = m_{paraxial}$$

Or, (Abbe's original formulation,[1]):

$$\frac{SinU'}{u'} = \frac{SinU}{u}$$

© 2025 Synopsys, Inc.

# What if the Abbe Sine Condition is Not Met?

(Reference 2)

- If the condition is not met, then the outer zones of the pupil see different magnifications than the paraxial zone.

- This is a description of the aberration known as coma:

$$W_{131} H \rho^3 \cos\phi$$

$H$ is the normalized field position
$\rho$ and $\phi$ are the coordinates of the ray in the pupil

which can be rewritten as:

$$W_{131}\, \rho^2 \, (H \rho \cos\phi)$$

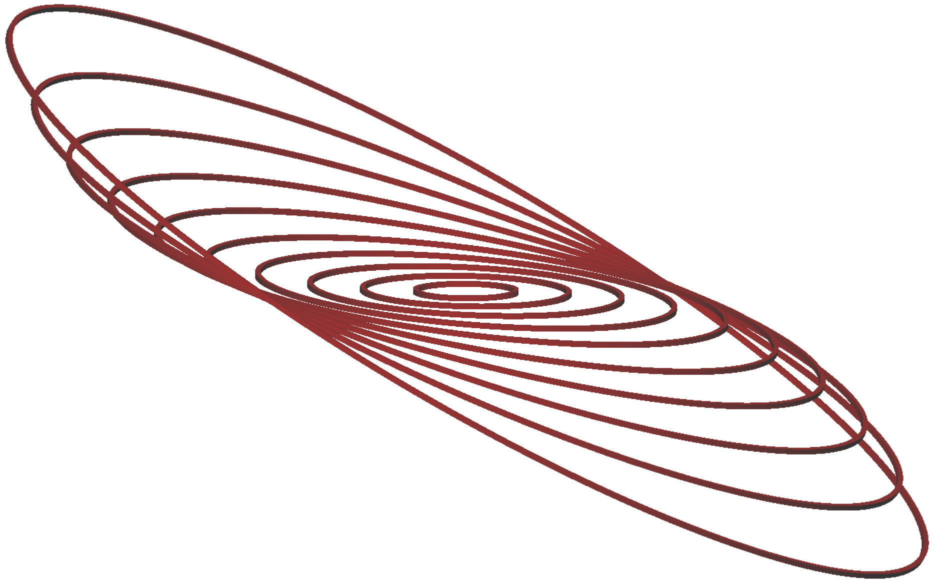
$\rho \cos\phi$ is a tilt of the wavefront

$H \rho \cos\phi$ is a tilt of the wavefront that increases linearly with field, i.e., a magnification change.

The $\rho^2$ term means that the magnification change increases quadratically with the radial position in the pupil.

© 2025 Synopsys, Inc.

# Comatic Wavefront



A series of concentric but increasingly tilted circles generates a comatic wavefront.


© 2025 Synopsys, Inc.

# Offense Against the Sine Condition

For the case of zero spherical aberration

© 2025 Synopsys, Inc.

# $Offense\ Against\ the\ Sine\ Condition\ (OSC)$

(Reference 3)

The Sine Condition is:

$$\frac{SinU'}{u'} = \frac{SinU}{u}$$

This can be rewritten as:

$$\frac{\frac{SinU}{u}}{\frac{SinU'}{u'}} = 1$$

Define OSC to be the error in that expression:

$$OSC = \frac{\frac{SinU}{u}}{\frac{SinU'}{u'}} - 1$$

This can be rewritten as:

$$OSC = \frac{\frac{nSinU}{n'SinU'}}{\frac{nu}{n'u'}} - 1 = \frac{M - m}{m}$$

…where M is the magnification at a particular zone and m is the paraxial magnification. Thus we see that OSC (for a particular zone) is numerically equal to the fractional error in the magnification for the zone in question.

© 2025 Synopsys, Inc.

# Numerical Implication of OSC

- OSC is equal the fractional error in the magnification for a particular zone of the pupil.
- For example, if the OSC for a particular zone of the pupil is 0.01, then the magnification for that zone of the pupil differs from the paraxial magnification by 1%.
- Roughly speaking, this means that:
  - If the chief ray behaves according to the magnification *m*, and the rays in the zone in question behaves according to the magnification *M* (which in this example differs from the paraxial magnification by 1%), then some measure of the blur will be equal to 1% of the image height.
  - The local image height grows linearly with field, so the blur also grows linearly with field.
- We have already recognized that the aberration is coma, and we now see specifically that it is coma that grows linearly with field (not, for instance, "field-cubed coma").
- The blur shape for coma is an "ice cream cone." What measure of that shape is equal to OSC times the image height?

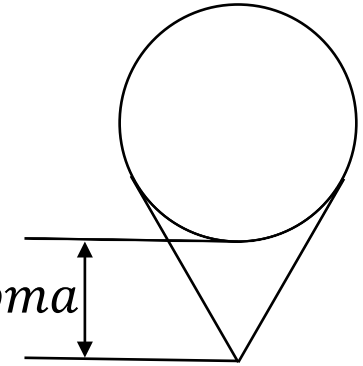
© 2025 Synopsys, Inc.

# More Precisely (For systems free of Spherical Aberration)

(Reference 3)

It can be shown that (for systems free of spherical aberration) OSC times the local field height is the quantity known as “sagittal coma”:

$$y'_{paraxial} * OSC = Sagittal\ Coma$$



That is, the OSC gives the sagittal coma, as a fraction of the image height:

$$OSC = \frac{Sagittal\ Coma}{y'_{paraxial}}$$

Roughly speaking, if the sagittal coma is 1% of the image height (0.5% of the image diameter, then the system cannot resolve more than 200 “pixels”.  Typically, OSC needs to be much better than this!

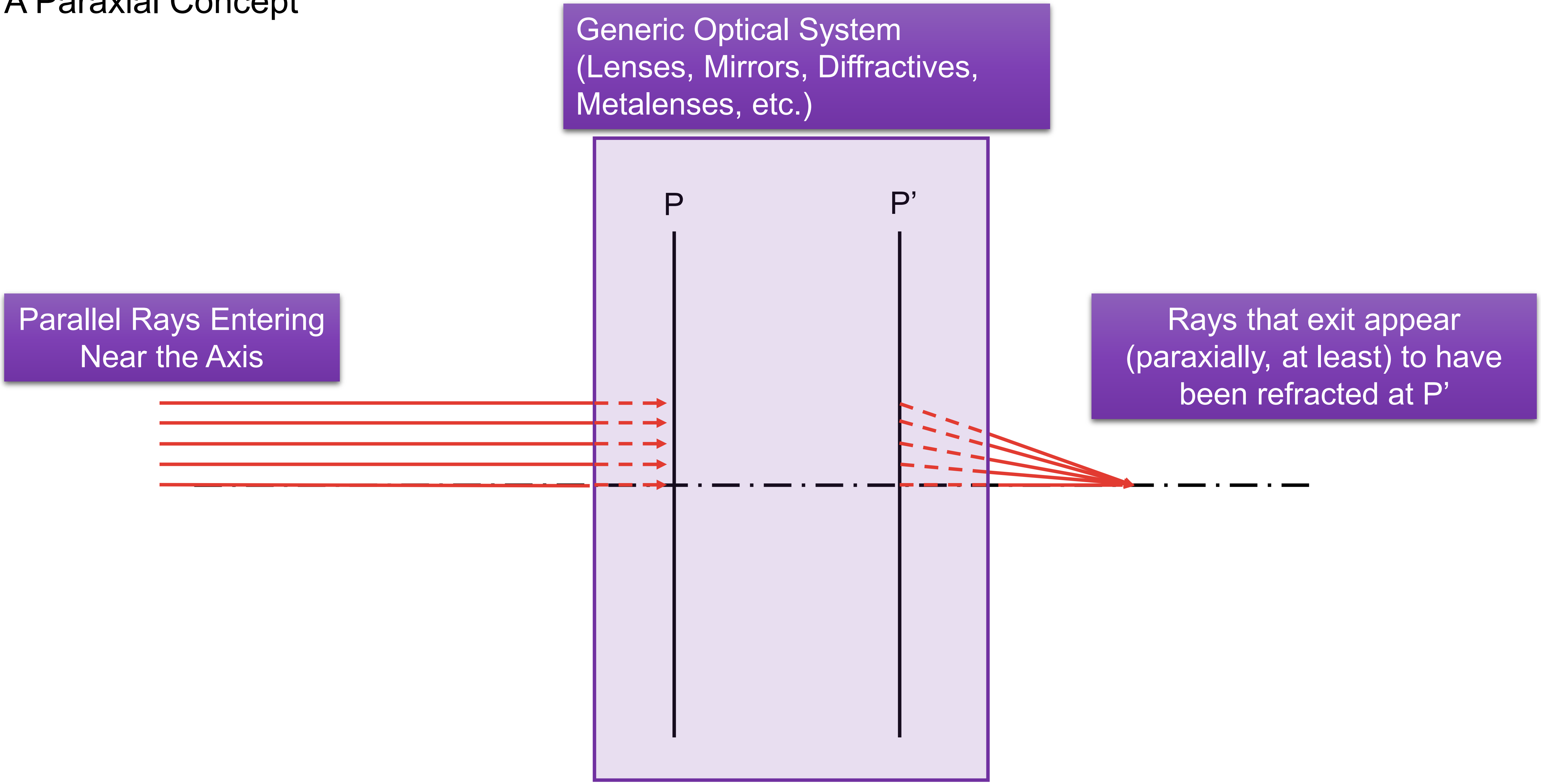
© 2025 Synopsys, Inc.

# Implications of the Abbe Sine Condition

© 2025 Synopsys, Inc.

# Principal Planes

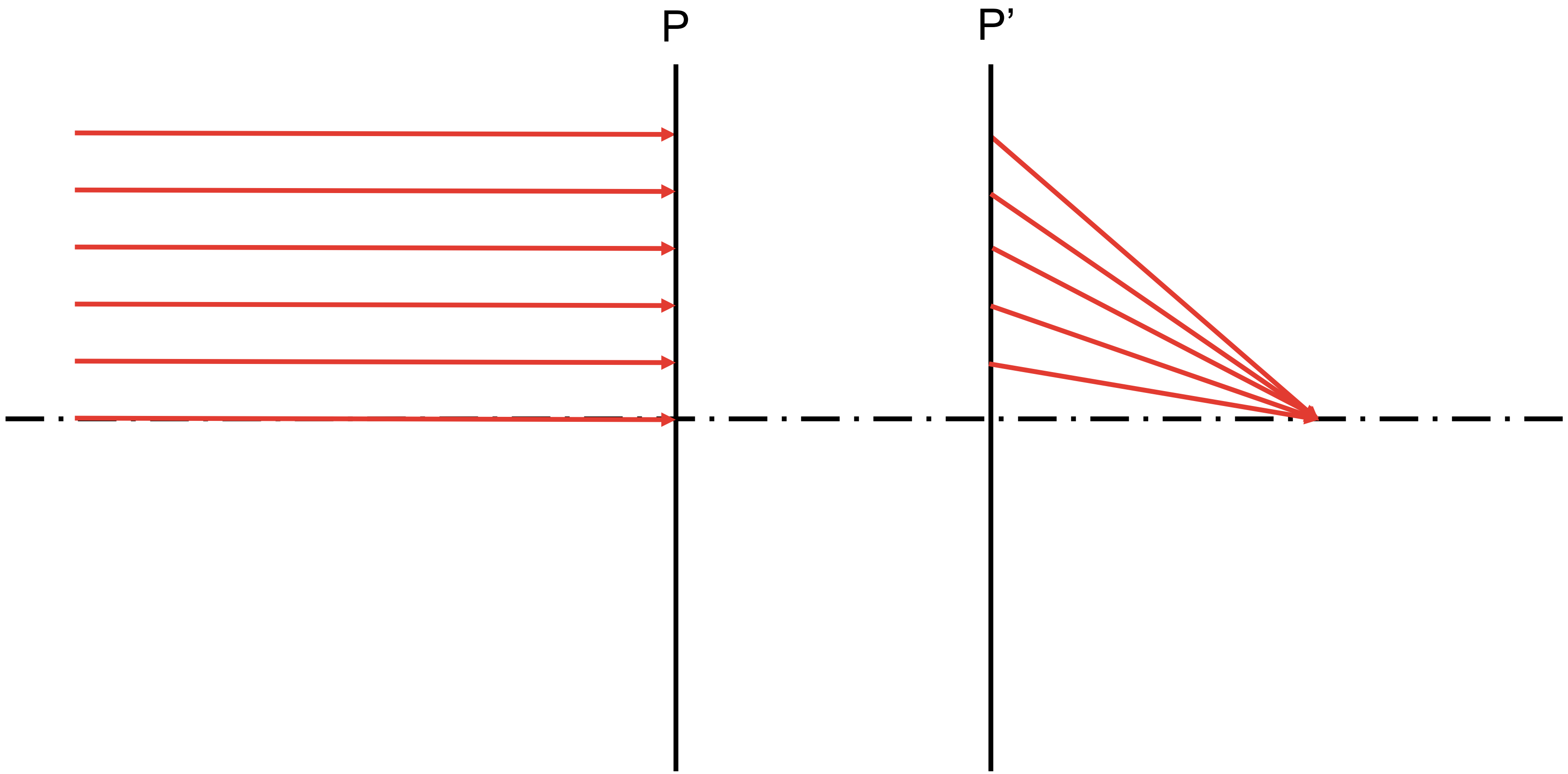
## A Paraxial Concept




© 2025 Synopsys, Inc.

# Refraction in the Non-Paraxial Region

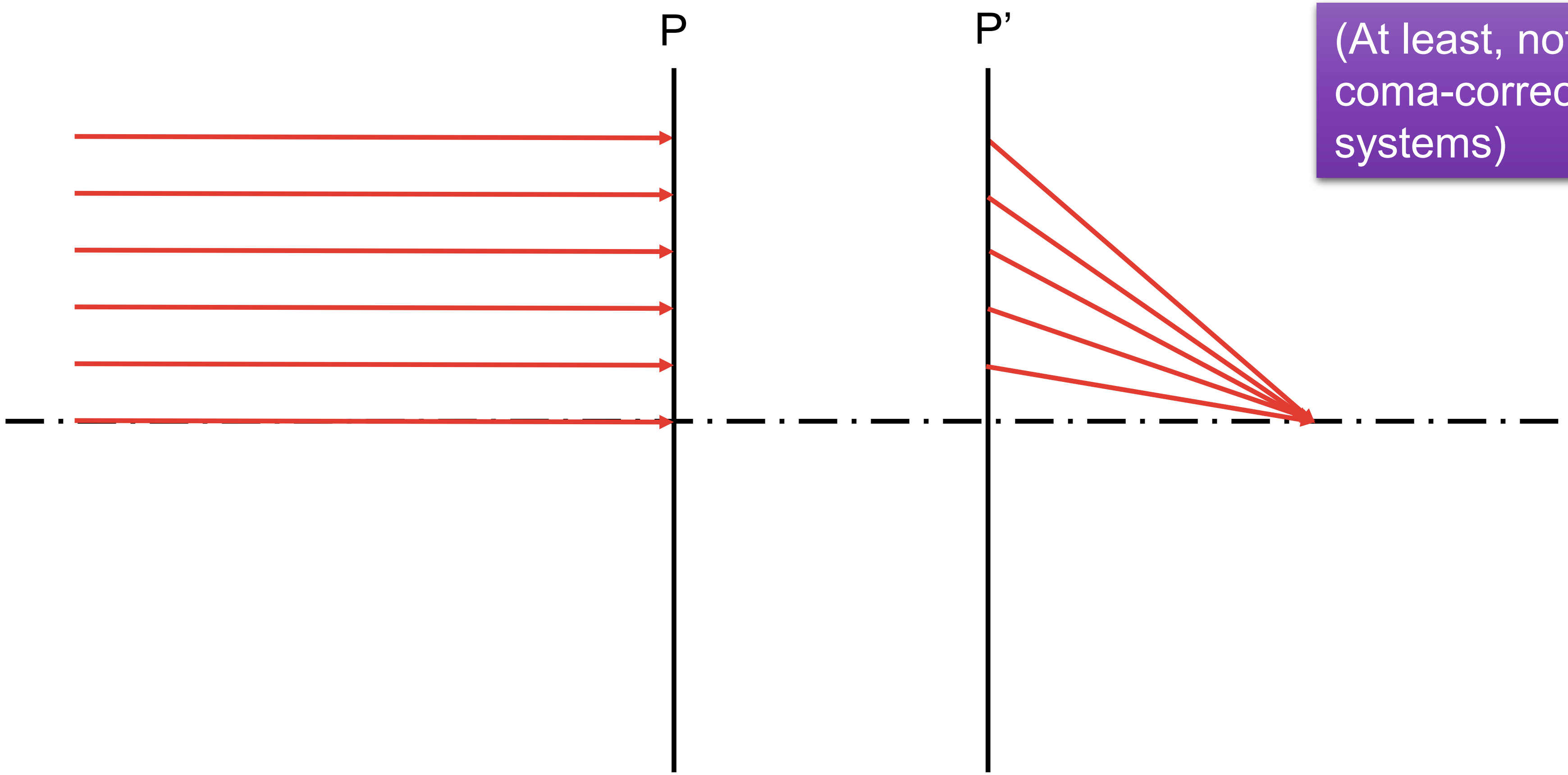
This is *NOT* how real optical systems work, in the extra-paraxial region!




© 2025 Synopsys, Inc. 

# Refraction in the Non-Paraxial Region

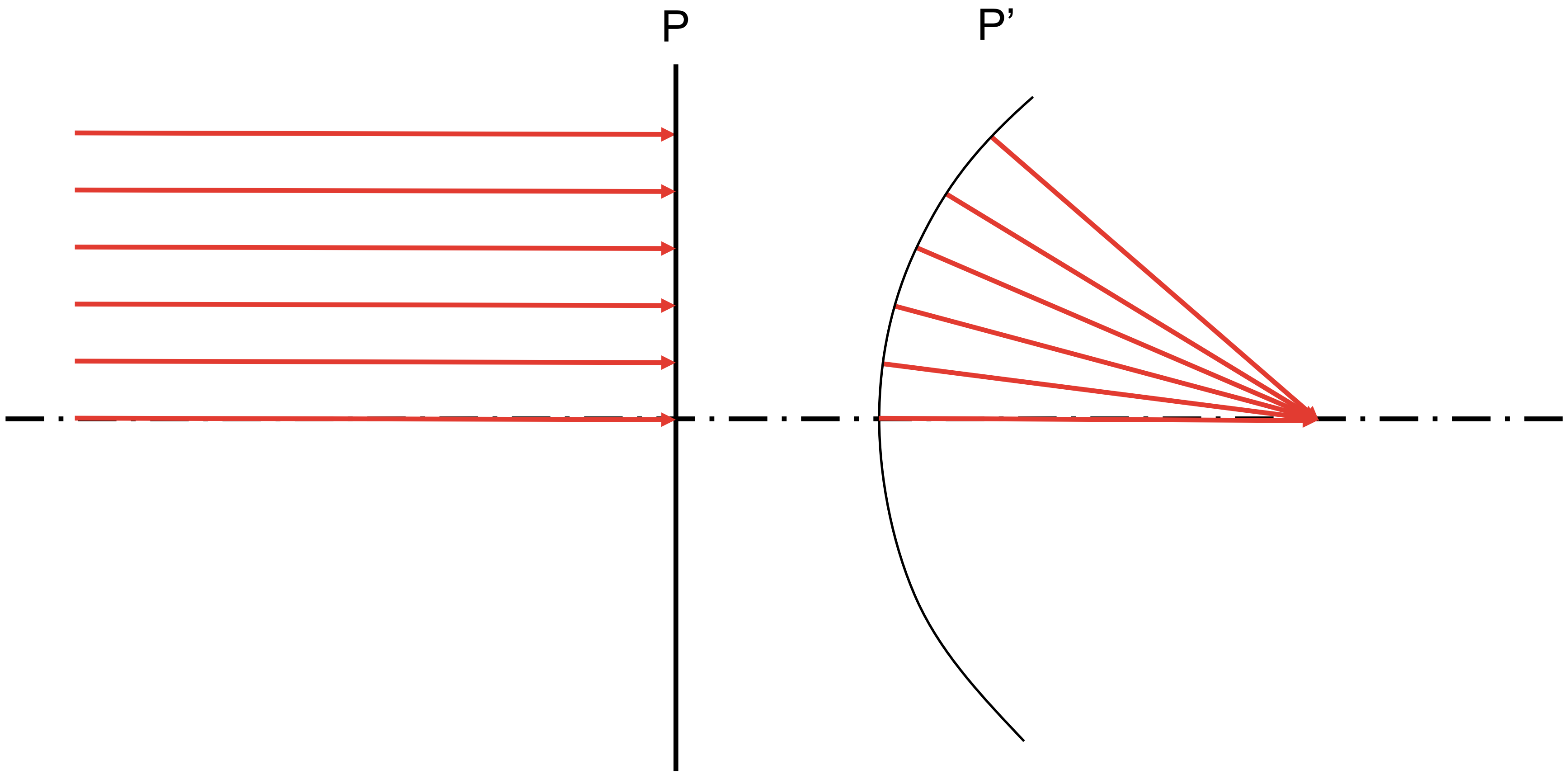
This is *NOT* how real optical systems work, in the extra-paraxial region!




(At least, not any coma-corrected systems)

# Refraction in the Non-Paraxial Region

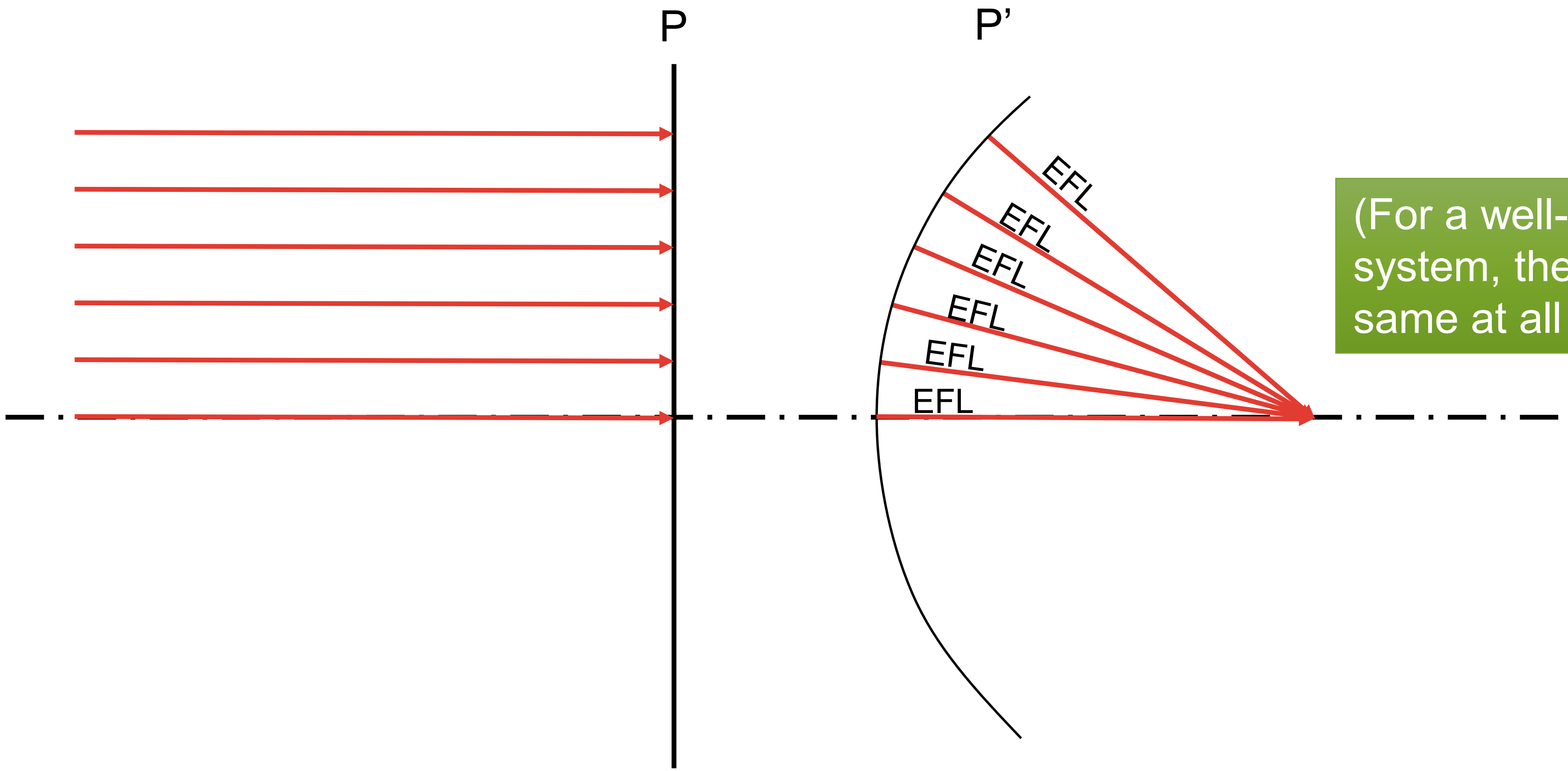
*THIS* is how real, coma-corrected systems work:




© 2025 Synopsys, Inc.

# How Focal Length is Measured

*THIS* is how the EFL is measured:




(For a well-designed system, the EFL is the same at all zones.

© 2025 Synopsys, Inc.

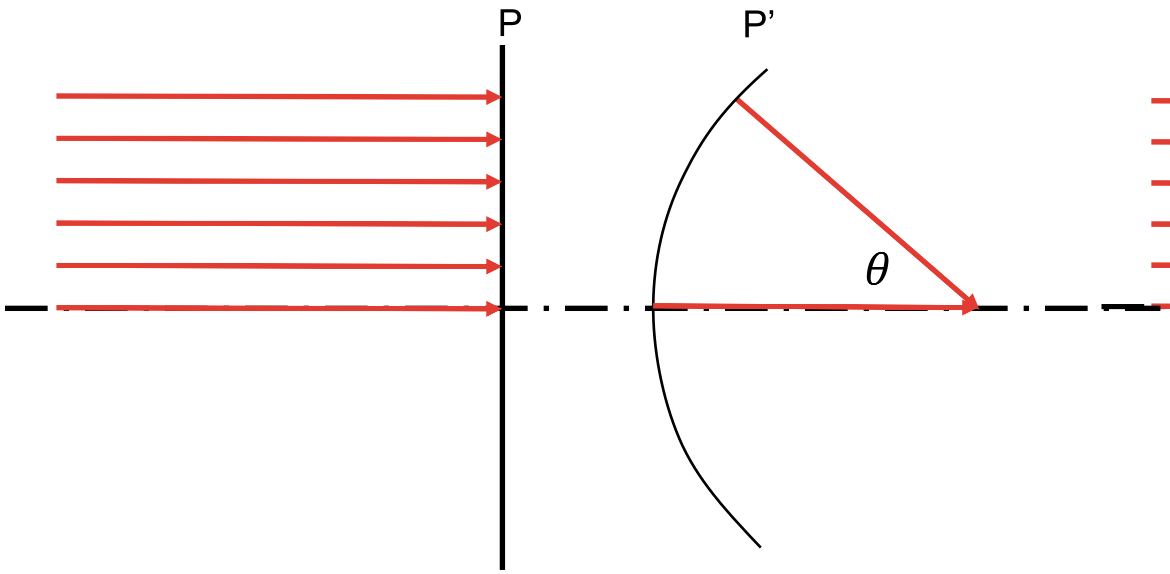
# Flat Optics

## Meta-Optics and

## Diffractive Optics (on planar substrates)

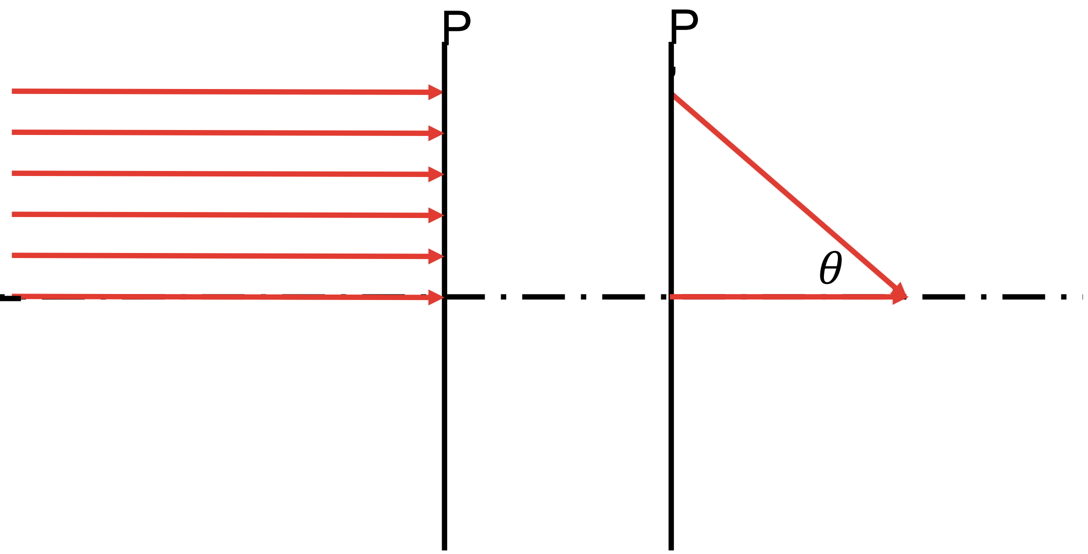
# Relationship between Ray Heights and Ray Angles

Coma Free System




$\sin U = Y/EFLmarginal$

Flat Optics System




$tanU = Y/EFLmarginal$

© 2025 Synopsys, Inc.

# Fractional error in (Flat Optic) focal length at the outer zone

$$EFLdesired = \frac{Y}{sinU}$$

$$EFLmarginal = \frac{Y}{tanU}$$

$$Fractional\ Error = FE = \frac{EFLmarginal - EFLdesired}{EFLdesired} = \frac{\frac{Y}{tanU} - \frac{Y}{sinU}}{\frac{Y}{sinU}}$$

$$FE = cosU - 1$$

(Depends only on the F-Number)

© 2025 Synopsys, Inc.

# Numerical Values For Flat Optics

$$FE = cosU - 1$$

For flat optics:

| F-Number | U (degrees) | CosU | Fractional Error |
|---|---|---|---|
| 4 | 7.125 | 0.9923 | -0.0077 |
| 2.8 | 10.125 | 0.9844 | -0.0156 |
| 2 | 14.036 | 0.9701 | -0.0299 |
| 1.4 | 19.654 | 0.9417 | -0.0583 |
| 1 | 26.565 | 0.8944 | -0.1055 |

At F/4, the coma is 0.8% of the field height;

at F/2, it is 3%, and

at F1 it is 10.6%

© 2025 Synopsys, Inc.

# Third- and Fifth-Order Coma

$$FE = cosU - 1$$

$$cosU \approx 1 - \frac{U^2}{2!} + \frac{U^4}{4!} - \cdots$$

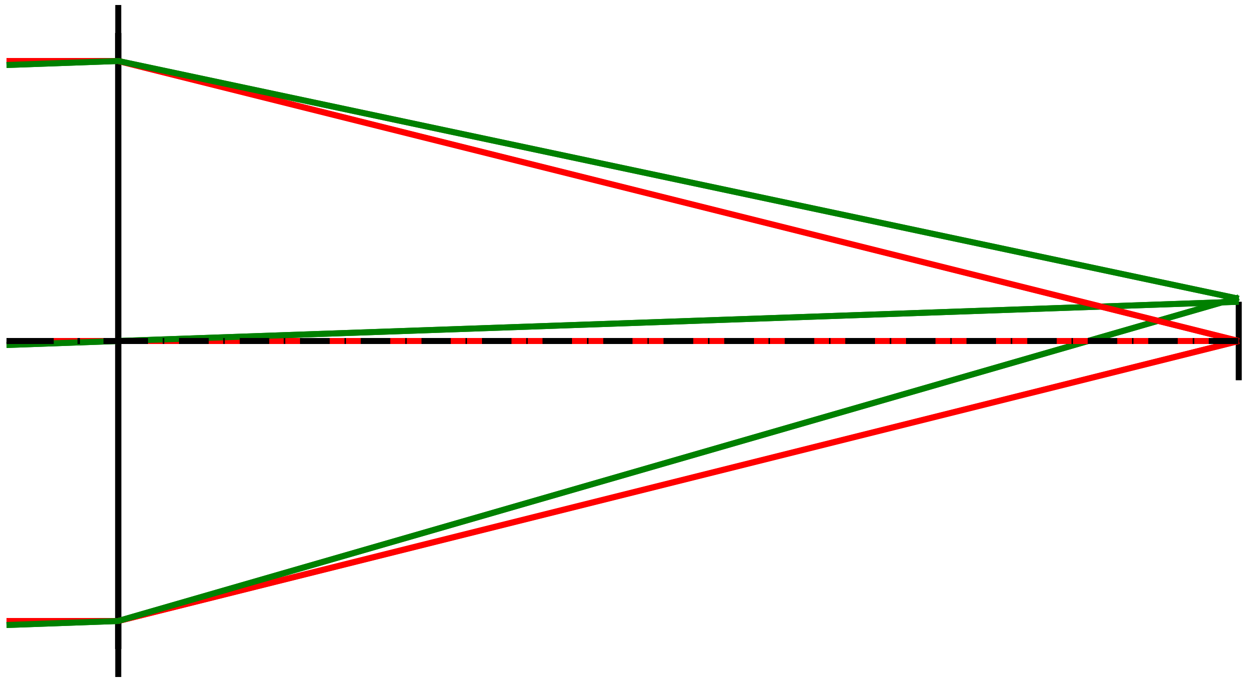
$$FE \approx -\frac{U^2}{2} + \frac{U^4}{24} - \cdots$$

$$W_{131}\,\rho^2\,(H\,\rho\,cos\phi\,) \qquad W_{151}\,\rho^4\,(H\,\rho\,cos\phi\,)$$

# Design Examples

# Planar Metalens

## Stop-At-Lens

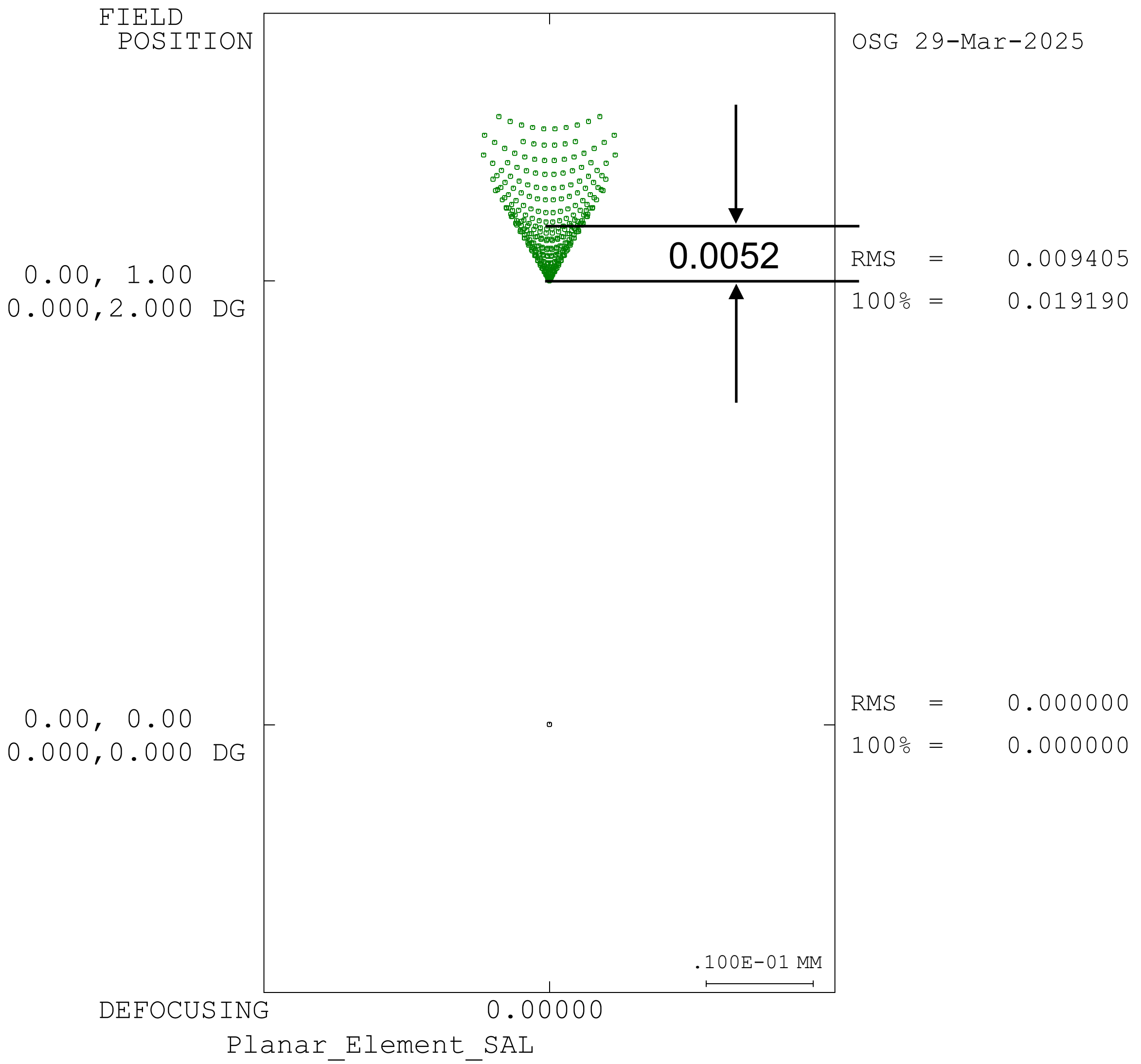


- EFL = 5
- F/2
- 587.6 nm
- Yan = 0, 2 deg.
- YIM = 0.1746 mm
- Expect Sag Coma to be 3% of 0.1746 = 0.005238

© 2025 Synopsys, Inc.

# Planar Element

## Stop-At-Lens

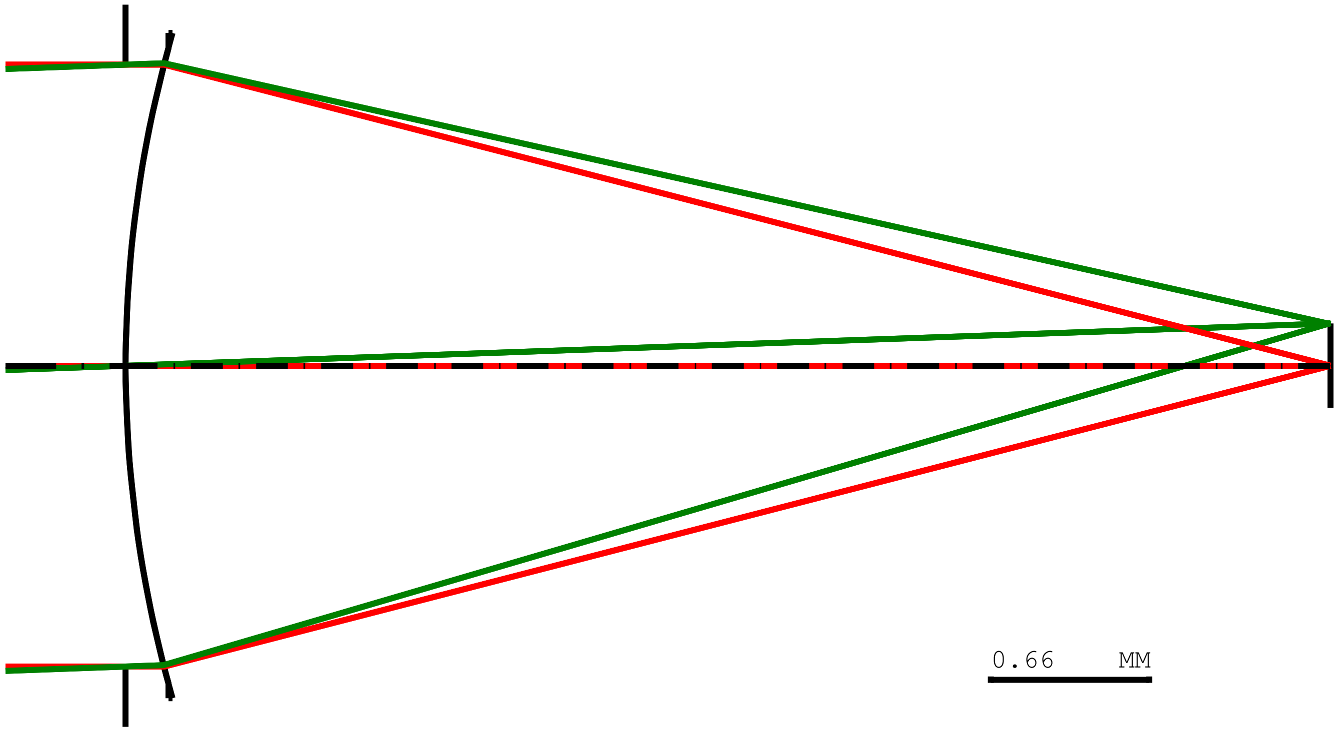



- Sagittal Coma is 0.0052 mm, as expected.

© 2025 Synopsys, Inc.

# Curved Substrate

## Stop-At-Lens

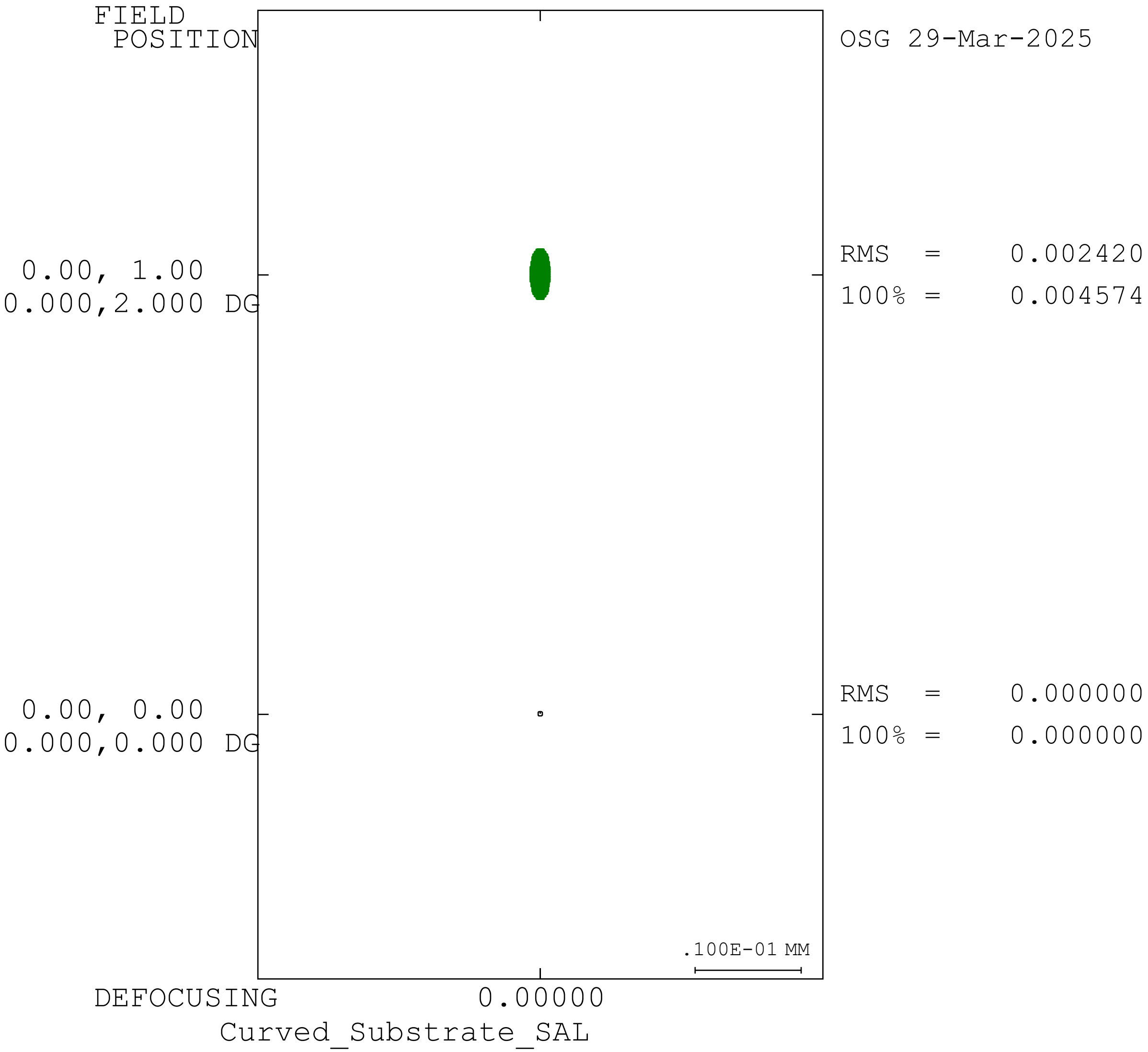



- EFL = 5
- Substrate Radius = 5
- F/2
- 587.6 nm
- Yan = 0, 2 deg.
- YIM = 0.1746 mm
- Expect Sag Coma to be Zero

© 2025 Synopsys, Inc.

# Curved Substrate

## Stop-At-Lens




Curving the substrate helps.

This is possible for diamond-turned diffractive elements on non-flat substrates.

# Seidel Aberration Theory

## Metalenses and Diffractive Elements on Planar Substrates


© 2025 Synopsys, Inc.

# “Designing Holographic Optical Elements”

## William Sweatt dissertation, U. of Arizona, 1977

- Sweatt’s thesis contained an aberration theory for diffractive optics, as follows:
  - Write out the Seidel theory for thin lenses (as above)
  - Take the limit as the index approaches infinity, with the power of each element remaining constant. For the power to remain constant as n→ ∞, the difference between the two surface curvatures must → $0$.
  - Chromatic aberration can be included in the model by making the index be proportional to the wavelength, e.g.
    - $n_{486.1\ nm} = 486.1\beta\ as\ \beta \to \infty$
    - $n_{587.6\ nm} = 587.6\beta\ as\ \beta \to \infty$
    - $n_{656.3\ nm} = 656.3\beta\ as\ \beta \to \infty$

- This “infinite index” approach was used in numerical simulation and optimization in the era before optical design codes had implemented “diffractive” as raytraceable surface types (late 1970’s and early 1980’s).

# One minor difficulty

- Sweatt's model works for flat substrates, and includes the possibility that the light sources used in the recording of holograms might be used off-axis.
- We would like to include the possibility that the substrate might be curved.
- This leads to a slightly modified set of equations.


© 2025 Synopsys, Inc.

# Wavefront Aberrations

## Hopkins, Wave Theory of Aberrations

$$W = W_{040}\rho^4 + W_{131}H\rho^3 cos\phi + W_{222}H^2\rho^2 cos^2\phi + W_{220}H^2\rho^2 + W_{311}H^3\rho cos\phi + \ldots$$

Spherical Aberration | Coma | Astigmatism | Field Curvature | Distortion

- $W_{040} = \left(\frac{1}{8}\right) S_1;$
- $W_{131} = \left(\frac{1}{2}\right) S_2;$
- $W_{222} = \left(\frac{1}{2}\right) S_3;$
- $W_{220} = \left(\frac{1}{4}\right) S_4;$
- $W_{311} = \left(\frac{1}{2}\right) S_5;$

$S_1 \ldots S_1$ are the “Seidel Aberrations”

We will list the Seidel aberrations for thin lenses in a few slides.

 © 2025 Synopsys, Inc. 

# The Aberration Theory of Thin Lenses

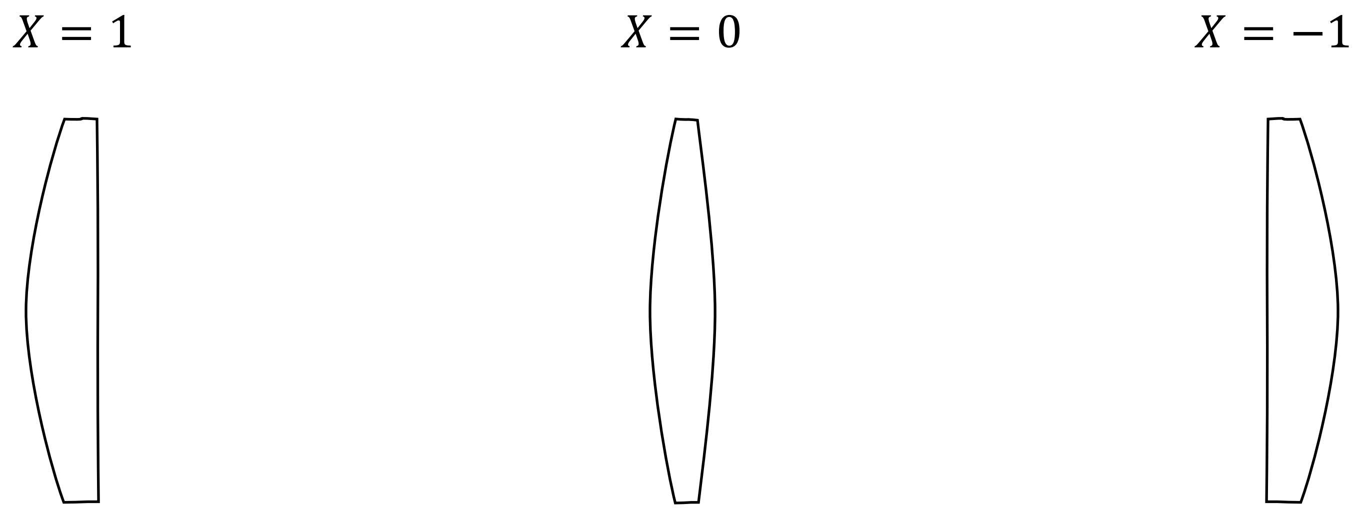
References: Kidger, Sasian, Hopkins

- The aberration theory for thin lenses has been published by various authors [2,4, and 5].
- The aberrations of the lenses are calculated in terms of several important parameters:
  - Ж, the Lagrange Invariant
  - $\varphi$, the lens power
  - n, the refractive index
  - X: the Bending Parameter
  - Y: the Conjugate Parameter
- The aberrations are initially calculated for the stop-at-lens case.
- Changes of the aberrations with stop position are considered separately (which is instructive).


© 2025 Synopsys, Inc.

# X: The Bending Parameter

Define: $$X \equiv \frac{c_1 + c_2}{c_1 - c_2}$$

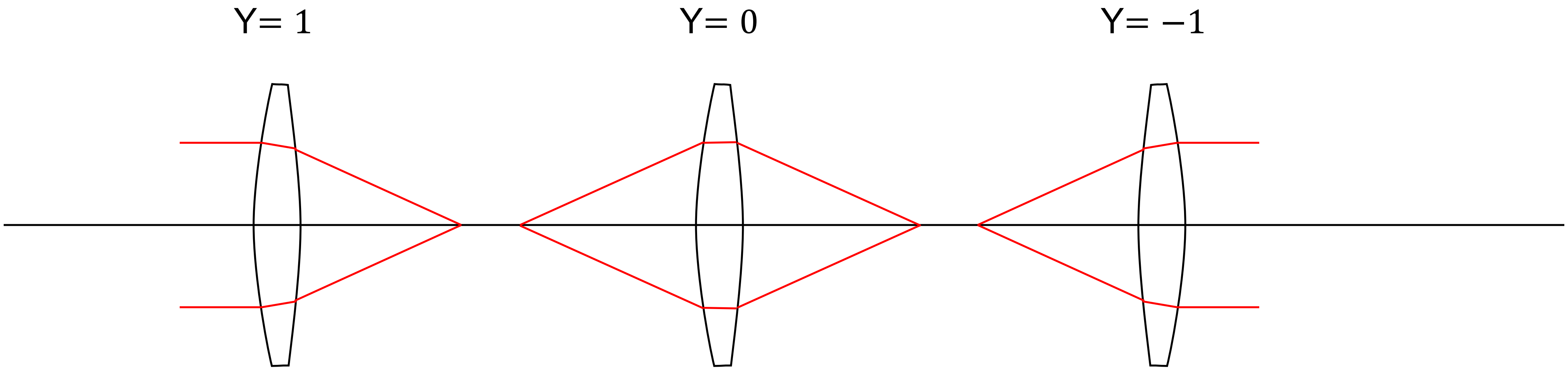




© 2025 Synopsys, Inc.

# Y: The Conjugate Parameter

Define: $Y \equiv \frac{u' + u}{u' - u} = \frac{1 + m}{1 - m}$ (Conjugate Factor)



# Aberrations of Thin Lenses (Stop-at-Lens)

Contribution from spherical surfaces

Aspheric Contribution

For the stop-at-lens case:

$$S_1 = \frac{1}{4}y^4\varphi^3\left\{\frac{n+2}{n(n-1)^2}X^2 - \frac{4(n+1)}{n(n-1)}XY + \frac{3n+2}{n}Y^2 + \frac{n^2}{(n-1)^2}\right\} + \alpha y^4$$

$$S_2 = \frac{1}{2}Жy^2\varphi^2\left\{\frac{n+1}{n(n-1)}X - \frac{2n+1}{n}Y\right\}$$

$$S_3 = Ж^2\varphi$$

$$S_4 = Ж^2\varphi\left(\frac{1}{n}\right)$$

$$S_5 = 0$$


© 2025 Synopsys, Inc.

# Surface Asphericities

- Classical aberration theory for thin lenses assumes (for historical manufacturing reasons) that the surfaces are spherical and applies a correction term in the event the surfaces are aspheric.
- We will return to this point later in the context of metalenses.
- Worth noting:
  – For the stop at the lens case, the asphericity only affects the spherical aberration, not the off-axis aberrations.

# Seidel Theory For Flat Optics

Apply Sweatt’s “Infinite Index” approach

 © 2025 Synopsys, Inc. 

# Curved Substrates are Problematic

For equi-convex or equi-concave lenses, the bending parameter $X \equiv \frac{c_1+c_2}{c_1-c_2}$ is zero and remains zero regardless of the index.

For asymmetric elements, X becomes problematic at high index:

- As the index increases, the two curvature values approach the same value if the power is held constant.
- The denominator approaches zero, but the numerator does not.

To avoid this problem, we will reformulate the equations in terms of the average curvature:

$$c = \frac{c_1 + c_2}{2}$$

This can be rewritten as:

$$c = \frac{X\varphi}{2(n-1)}$$

From which we have:

$$\mathrm{X} = \frac{2c(n-1)}{\varphi}$$


© 2025 Synopsys, Inc.

# Seidel Aberrations of Thin Lenses (Stop-at-Lens)

We have:

$$X = \frac{2c(n-1)}{\phi}$$

Substituting into our previous expressions and taking the high index limit, we obtain:

$$S_1 = \frac{1}{4} y^4 \varphi (4c^2 - 8\varphi c Y + 3\varphi^2 Y^2 + \varphi^2 + 4\alpha)$$

$$S_2 = Ж y^2 \varphi (c - \varphi Y)$$

$$S_3 = Ж^2 \varphi$$

$$S_4 = 0$$

$$S_5 = 0$$

 © 2025 Synopsys, Inc. 

# Spherical Aberration

(Stop-at-Lens Case)

$$S_1 = \frac{1}{4} y^4 \varphi \{4c^2 - 8\varphi c Y + 3\varphi^2 Y^2 + \varphi^2 + 4\alpha\}$$

In the context of a diffractive and metalens design, it is worth noting that:

- The asphericity parameter $\alpha$ may be chosen to make the spherical aberration vanish for any desired combination of $c, \varphi,$ and $Y$.
- This happens automatically if a holographic element is photographically recorded using two point-sources of light at the same wavelength as the reconstruction. (One can say that the diffractive element “self-aspherizes” for the recording conditions.)
- The spherical aberration will change if either the conjugate (Y) or the substrate curvature changes.


© 2025 Synopsys, Inc.

# Coma

(Stop-at-Lens Case)

$S_2 = Жy^2\varphi\{c - \varphi Y\}$

- For any conjugate $Y$ and lens power $\varphi$, the substrate curvature $c$ may be chosen to eliminate coma.
  - For the case of parallel rays entering the lens $(Y = 1)$, the solution is $c = \varphi$, i.e., the substrate is concentric with the image. (This agrees with the Abbe Sine Condition.)
  - For the case of symmetrical 1:1 imaging, $(Y = 0)$, the solution is $c = 0$, i.e., a planar substrate. (This is the only condition under which coma vanishes for a planar substrate.)
  - For the case of a point source being collimated by the lens $(Y = -1)$, the solution is $c = -\varphi$, i.e., the substrate is concentric with the object. (This agrees with the Abbe Sine Condition.)

 © 2025 Synopsys, Inc. 

# Astigmatism

## (Stop-at-Lens Case)

$S_3 = Ж^2 \varphi$

- Astigmatism does not depend on the conjugate parameter $(Y)$ or the substrate curvature.
- Except in the trivial cases that $\varphi = 0$ or $Ж = 0$, Astigmatism is non-zero for the stop-at-lens case, and nothing can be done about it.
- (We will discuss stop-shift effects later)


© 2025 Synopsys, Inc.

# Field Curvature

(Stop-at-Lens Case)

$$S_4 = 0$$

- Field curvature is exactly zero for diffractive and metalens elements.
  - This is a big advantage if all or most of the system power is diffractive or metalens in origin.
  - This can be a *dis*advantage in the case of hybrid system, as it is not possible to use diffractive or metalens elements to correct for the field curvature of refractive lenses.


© 2025 Synopsys, Inc.

# Distortion

(Stop-at-Lens Case)

$$S_5 = 0$$

- Field curvature is exactly zero for the stop-at-lens case.
- (This is also true of refractive lenses.)


© 2025 Synopsys, Inc.

# Stop Shift Equations
# (How the Seidel Sums change with Stop Shift)

Define the “Eccentricity” for each surface:

$$E_s \equiv \left(\frac{\bar{y}_s}{y_s}\right);$$

Ratio of the chief ray height to the marginal ray height for surface *s*.

- A measure of how far an element is from the stop.
- $E$ varies from element to element within the system.

How it varies with stop shift:

$$\delta E \equiv \left(\frac{\delta \bar{y}}{y}\right)$$

Change of E with stop position.

$\delta E$ is constant through a system and can be measured at any surface except at an image.


© 2025 Synopsys, Inc.

# Stop Shift Equations
# (How the Seidel Sums change with Stop Shift)

Define:

$$E \equiv \left(\frac{\bar{y}}{y}\right); \quad \delta E \equiv \left(\frac{\delta \bar{y}}{y}\right)$$

Then

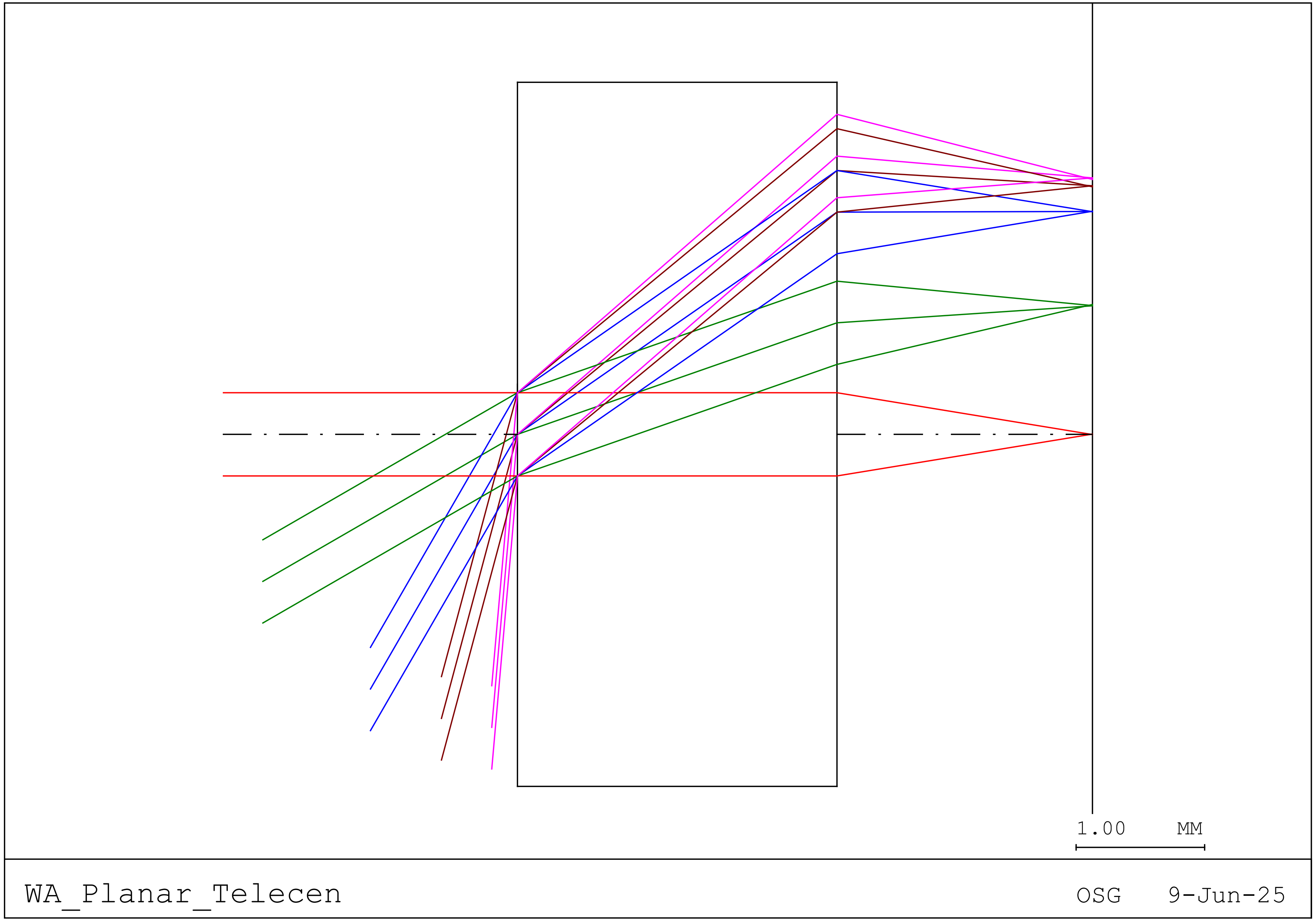
$$\delta S_1 = 0$$
$$\delta S_2 = (\delta E) S_1$$
$$\delta S_3 = (\delta E)^2 S_1 + 2(\delta E) S_2$$
$$\delta S_4 = 0$$
$$\delta S_5 = (\delta E)^3 S_1 + 3(\delta E)^2 S_2 + (\delta E)(S_4 + 3S_3)$$

We can influence the coma if $S_1 \neq 0$.

Astigmatism varies with stop position if either $S_1 \neq 0$ or $S_2 \neq 0$


© 2025 Synopsys, Inc.

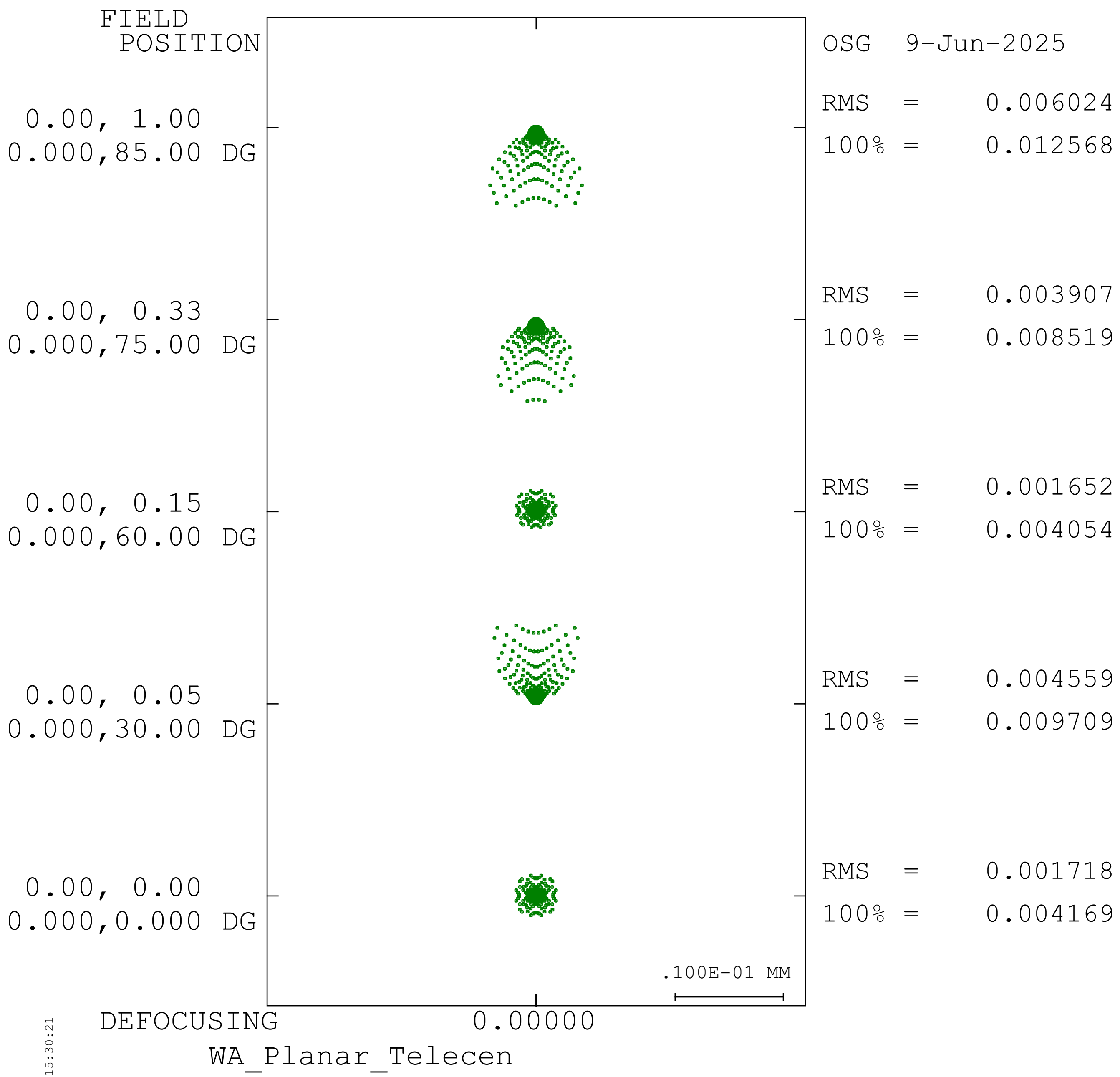
# A Wide-Angle Metalens




- A metalens covering a semi-field of 85º at F/3.
- Similar to the design of Fan, Lin, and Su (Reference 6).
- The focal length is 2 mm.

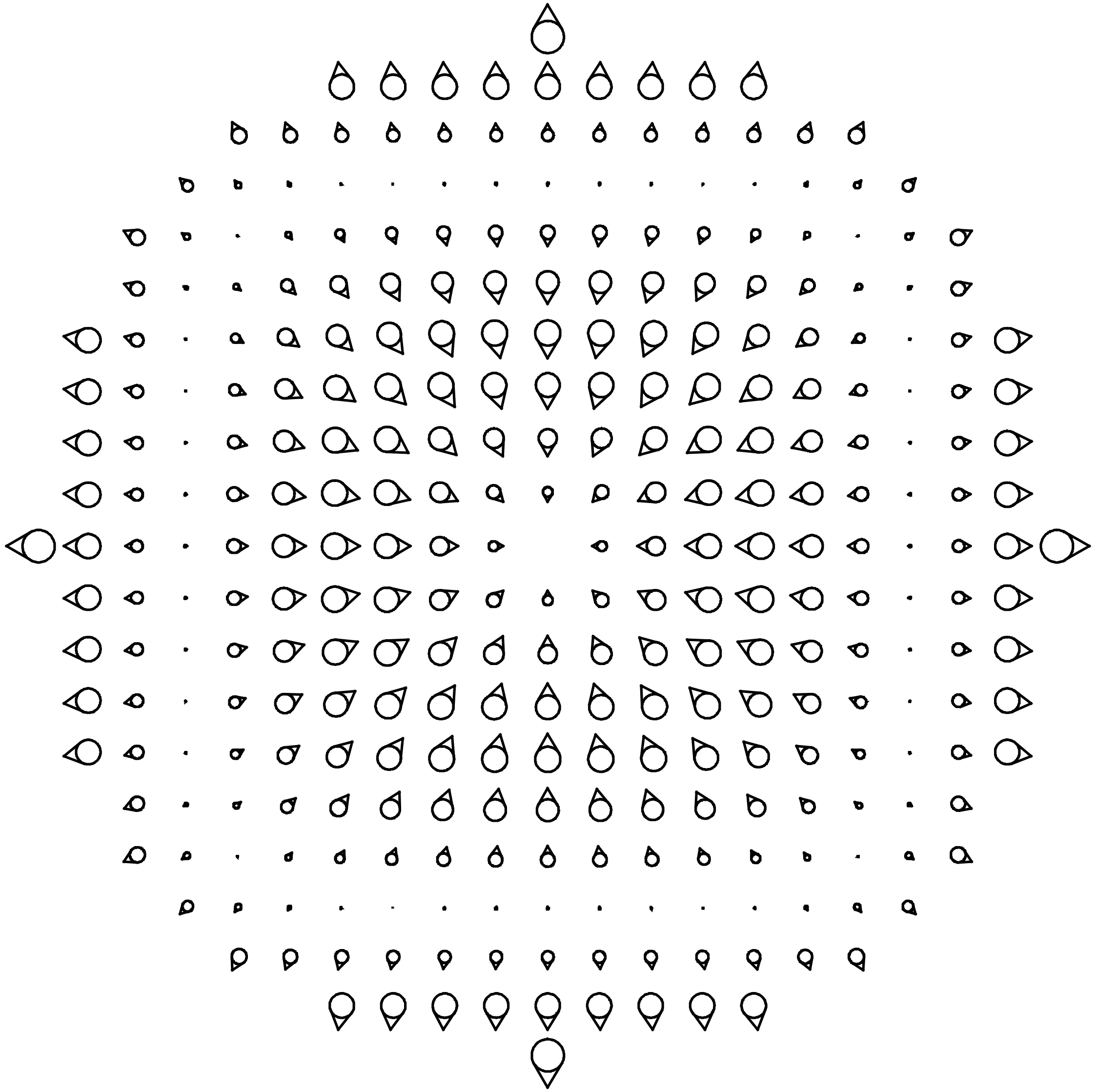
© 2025 Synopsys, Inc.

# Wide-Angle Metalens




- Field-linear and field-cubed coma are both present in this design.
- A small amount of spherical aberration is allowed on axis, in order that coma is affected by the stop position.
- Coma is balanced and reverses sign at the 60º point.

# Wide-Angle Metalens



- Field-linear and field-cubed coma are both present in this design.
- A small amount of spherical aberration is allowed on axis, in order that coma is affected by the stop position.
- Coma is balanced and reverses sign half-way out in the field.

© 2025 Synopsys, Inc.

# References


1. Abbe, E., Hon. (1881), VII.—*On the Estimation of Aperture in the Microscope.*. Journal of the Royal Microscopical Society, 1: 388-423. https://doi.org/10.1111/j.1365-2818.1881.tb05909.

2. H. H. Hopkins, **Wave Theory of Aberrations**, Clarendon Press, Oxford, (1950)

3. A. Cox, **A System of Optical Design**, Focal Press, New York, (1964)

4. M. J. Kidger, **Fundamental Optical Design** SPIE Press, Bellingham WA (2002)

5. J. Sasian, **Introduction to Lens Design**, Cambridge University Press, Cambridge, (2019)

6. Fan, CY., Lin, CP. & Su, GD.J. "Ultrawide-angle and high-efficiency metalens in hexagonal arrangement." *Sci Rep* **10**, 15677 (2020). https://doi.org/10.1038/s41598-020-72668-2

7. L. von Seidel "Zur Dioptrik. Über die Entwicklung der Glieder 3ter Ordnung welche den Weg eines ausserhalb der Ebene der Axe gelegene Lichtstrahles durch ein System brechender Medien bestimmen." Astronomische Nachrichten V. 1027, 1028, 1029.


© 2025 Synopsys, Inc.

# Thank you